

Rational curves and isoparametric transformations

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Abstract: Curve approximation associated with the finite element method usually implies linear or parabolic approximating segments when the transformation of polygonal master-elements is involved. We consider the construction of transformations and of associated bases that result in general conic approximating curve segments, while still allowing us to do all the required calculations on the simpler straight-edged elements. We show that projective transformations can be used to produce conic parameterizations in a systematic way. Examples of transformations and of suitable bases are given for triangular elements with one conic and two straight edges.

1. Introduction

The most popular plane curves used in computer-aided design are undoubtedly parametric splines—usually polynomial cubics, but occasionally also rational cubics [1,2]. The use of conics is of practical importance in, for example, mechanical engineering and the aircraft industry. All these curves are rational, that is, the homogeneous or projective coordinates of any point (X, Y, Z) on the curve can be expressed as polynomials of a parameter t , say, namely

$$X = \sum_{i=0}^n a_i t^i, \quad Y = \sum_{i=0}^n b_i t^i, \quad Z = \sum_{i=0}^n c_i t^i. \quad (1.1)$$

The line $Z = 0$ denotes the ideal line or line at infinity, and the Cartesian coordinates are $(x = X/Z, y = Y/Z)$ for $Z \neq 0$. The parameter t may be expressed as a point on a projective line if required, in which case the parameterization (1.1) consists of homogeneous polynomials, but we shall assume that we are interested in finite parameter values only. The algebraic curves (1.1) are generally of order n , although not all algebraic curves of order n can be expressed in the form (1.1). For example, a cubic with no double point cannot be rational [3]. Curves with polynomial parameterization, that is, with $Z = 1$, form a subset of the rational curves. For $n = 2$, $Z = 1$, the coordinates (1.1) represent points on parabolae or lines, but never ellipses or hyperbolae, while for $n = 3$, $Z = 1$, we have curves belonging to one of four affine equivalence classes, namely cubics with one or two points of inflection, or a node, or a cusp [4].

For finite element applications the method of construction of curved elements implies the geometry. It is desirable to have correspondence between the geometry of the design and of the analysis. To simplify calculations such as the integrations needed for the element stiffness matrices [5], it is convenient to construct curved elements by the transformation of straight-edged master-elements. The curves relevant to the analysis are now implied by the transformations. For isoparametric transformations [5] a basis defined on the master-element is used to define the transformation as well as to approximate the solution. Let $\{W_i\}$ be a biorthonormal Lagrange interpolation basis for a master-element in the (p, q) -plane that can, at the very least, interpolate constants exactly, that is,

$$1 = \sum_i W_i(p, q). \quad (1.2)$$

Define

$$x = \sum_i x_i W_i(p, q), \quad y = \sum_i y_i W_i(p, q), \quad (1.3)$$

where the pair of transformation parameters (x_i, y_i) denotes the image under the transformation (1.3) of the node (p_i, q_i) associated with $W_i(p, q)$. The images of lines, and therefore of the master-element edges, will have parameterizations consistent with the form of the basis functions. For example, for polynomial second degree basis functions, the straight edges will be mapped onto segments of parabolae or lines.

It is of course possible to select a basis to be used in the transformation (1.3) purely to obtain the geometry required of the image element. The construction of such bases has been discussed by Wachspress [1,6]. If necessary, such an element may be considered as a macro-element, and subdivision into smaller elements, each with its own basis, may be needed. The transformation (1.3) is no longer isoparametric, but rather parametric.

We shall restrict the discussion to triangular elements with one general conic edge and two straight edges, and with normalized local coordinates introduced in such a way that the straight edges lie on the axes, with vertices at $(0, 0)$, $(1, 0)$ and $(0, 1)$. We first consider a systematic way of obtaining suitable parameterizations of general conics. The next step is the construction of a parametric transformation of the form (1.3) in such a way that the conic segment is the image of the line segment connecting $(1, 0)$ and $(0, 1)$ in the affine (p, q) -plane. Finally, basis functions are associated with the transformation to complete the analogy with parametric transformations. The process therefore reverses the order of the usual parametric transformation where one starts with a given basis and subsequently obtains parameterizations of line segment images.

2. Projective geometry and conics

Algebraic curves are defined by homogeneous polynomials over an algebraically closed field such as the complex numbers [3]. Real algebraic curves have real coefficients and are represented by real points. We are considering planar conics represented by points that satisfy

$$P_2(X, Y, Z) = A_{20}X^2 + A_{11}XY + A_{02}Y^2 + A_{10}XZ + A_{01}YZ + A_{00}Z^2 = 0.$$

The point with homogeneous coordinates (X, Y, Z) lies in the projective plane, so that we can accommodate points on the ideal line $(Z = 0)$ as easily as affine points $(Z \neq 0, \text{ equivalently } Z = 1)$.

Projective geometry is concerned with properties invariant under projective transformations [7], that is, under the multiplication of the position vector (X, Y, Z) by non-singular matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \quad (2.1)$$

Affine geometry is restricted to invariance under those projective transformations that keep a specific projective line (the ideal line) invariant, say the line $Z = 0$, in which case the affine transformations are characterized by the non-singular matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.2)$$

Note that the invariance of the ideal line is not a pointwise invariance.

The fundamental theorem of projective geometry [7] states that any four points, no three of which are collinear, can be mapped onto any four points, no three of which are collinear, by a unique (in the projective sense) projective transformation. This implies the projective equivalence of all projective planes. Similarly, any three non-collinear points can be mapped onto any three non-collinear points by an affine transformation. Furthermore, the three classes of conics distinguishable in an affine plane, namely ellipses, parabolae and hyperbolae, are invariant under affine transformations, since the ideal line is invariant and the conic properties on this line define the type. In the projective plane, however, all non-degenerate conics are projectively equivalent, that is, there exists a projective transformation that will map any given conic onto any other given conic. These transformations need not be unique. For example, the transformations

$$\begin{bmatrix} 0 & 0 & \kappa \\ 0 & \kappa^2 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

for all $\kappa \neq 0$, map the hyperbola $xy - 1 = 0$ onto the parabola $x^2 - y = 0$.

Note that the selection of four points on conic C_1 and four points on conic C_2 defines a unique projective transformation for the two sets of points, but this transformation will not in general map C_1 onto C_2 , although of course C_1 is mapped to a conic through the selected four points on C_2 . We therefore have to exercise some care in the construction of transformations. We want to construct a projective transformation for any given conic through $(1, 0)$ and $(0, 1)$ in the affine (x, y) -plane, such that the conic can be identified with the line segment

$$p = t, \quad q = 1 - t, \quad 0 \leq t \leq 1, \quad (2.3)$$

and we thus want to obtain a rational parameterization of the conic in a systematic way consistent with the intended finite element transformation.

3. Parameterization of general conics

We shall identify the general conic by means of a projective transformation with a specific, selected conic. The parameterization of this conic is then transformed inversely to find the associated parameterization of the given conic. Since we know that for a parabola we can find a

polynomial parameterization, we would like the obtained parameterization to be polynomial when the given conic is a parabola. Also, we should choose the special selected reference conic to be a parabola, which can then be associated with the segment (2.3) by means of the quadratic isoparametric transformation [8]

$$\xi = p(1 + \alpha q), \quad \eta = q(1 + \beta p), \quad (3.1)$$

where $\alpha = 2(2\xi_4 - 1)$, $\beta = 2(2\eta_4 - 1)$, and (ξ_4, η_4) is the image of the node $(\frac{1}{2}, \frac{1}{2})$ under the transformation (3.1). The transformation maps the standard triangle (Fig. 1(a)) onto the element with a parabolic curved edge through $(1, 0)$, (ξ_4, η_4) and $(0, 1)$ (Fig. 1(b)). Since we have $\beta\xi - \alpha\eta = \beta p - \alpha q$, the transformation (3.1) is a projection of points on (2.3) along parallel lines with slope β/α onto the parabolic segment

$$\xi = t(1 + \alpha - \alpha t), \quad \eta = (1 - t)(1 + \beta t), \quad 0 \leq t \leq 1. \quad (3.2)$$

The direction of the projection denotes the axial slope of the parabola. The use of projections along parallel lines to construct parameterizations is feasible only for parabolae (one possible direction) and hyperbolae (two directions). Projections along lines through a selected point on the conic may fail for particular point-conic combinations. We therefore construct parameterizations by means of projective transformations.

As the reference parabola to be mapped onto any given conic through $(1, 0)$ and $(0, 1)$ we select (3.2) with $\alpha = \beta = 1$ ($\xi_4 = \eta_4 = \frac{3}{4}$) so that the tangents at the vertices are parallel to the axes (Fig. 2(b)). The parameterization (3.2) becomes

$$\xi = t(2 - t), \quad \eta = 1 - t^2, \quad 0 \leq t \leq 1. \quad (3.3)$$

Any given conic through two points is completely defined by the intersection of the tangents at these two points together with any one other point on the conic [2]. Since the tangents may be parallel, let their point of intersection have homogeneous coordinates (X_t, Y_t, Z_t) and let (X_4, Y_4, Z_4) be an arbitrary point on the conic (Fig. 2(c)). Although $Z_4 = 1$ may be assumed, this results in some loss of symmetry in the resulting equations, so that we prefer to retain Z_4 as an arbitrary constant.

There is a unique projective transformation that maps the four points $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 1)$ and $(3, 3, 4)$ related to the reference parabola onto the corresponding four points $(0, 1, 1)$, $(1, 0, 1)$, (X_t, Y_t, Z_t) and (X_4, Y_4, Z_4) related to the given conic. The point $(\frac{3}{4}, \frac{3}{4})$ was chosen as the definitive point on the reference parabola. Since in each case the particular set of four

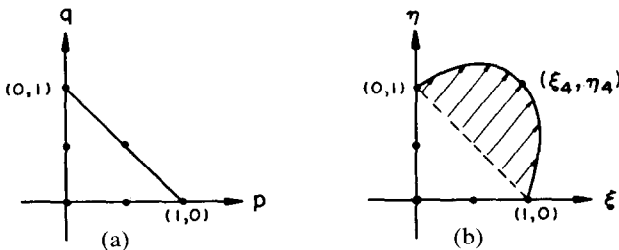


Fig. 1. Quadratic isoparametric transformation.

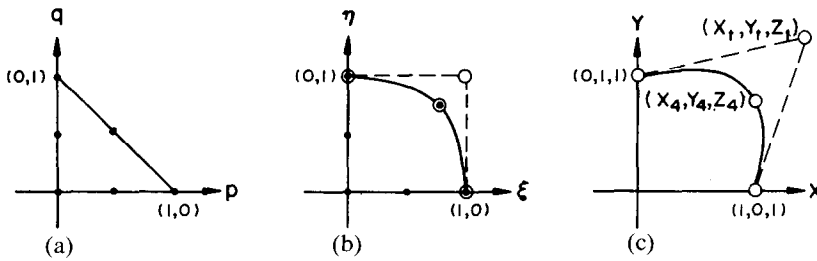


Fig. 2. Transformation sequence (a) to (b): quadratic isoparametric; (b) to (c): projective.

points defines a unique conic, the transformation must map the reference parabola onto the given conic. For the selected points the transformation is

$$\begin{bmatrix} cX_t & 3cX_t - 2kX_4 & -3cX_t + 2kX_4 \\ 3cY_t - 2kY_4 & cY_t & -3cY_t + 2kY_4 \\ 2cY_t + cZ_t - 2kY_4 & 2cX_t + cZ_t - 2kX_4 & -3cZ_t + 2kZ_4 \end{bmatrix} \quad (3.4)$$

subject to

$$k(X_4 + Y_4 - Z_4) = c(X_t + Y_t - Z_t), \quad ck \neq 0. \quad (3.5)$$

In terms of the parameter t of the line segment (2.3) and, by means of the isoparametric transformation, of the reference parabola (3.3), we therefore have the general parameterization for a given conic

$$\left. \begin{aligned} X &= t[cX_t + (kX_4 - 2cX_t)t], \\ Y &= (1-t)[kY_4 - cY_t - (kY_4 - 2cY_t)t], \\ Z &= (kZ_4 - 2cZ_t)t^2 - (2kY_4 - 2cY_t - cZ_t)t + kY_4 - cY_t, \end{aligned} \right\} \quad (3.6)$$

where $0 \leq t \leq 1$ denotes the relevant segment, and where (3.6) is still subject to the relation (3.5).

Note that the transformation (3.4) does not keep the origin invariant. This can, if required, be achieved by using an additional affine transformation as well as a different isoparametric transformation, but this seems pointless, since such a sequence of transformations will not in general keep the straight edges pointwise invariant—a property required for conformity with neighbouring elements.

The transformation (3.4) is affine, and the parameterization (3.6) polynomial, if and only if

$$(X_4, Y_4, Z_4) = (2X_t + Z_t, 2Y_t + Z_t, 4Z_t), \quad (3.7)$$

where equivalence is in the projective sense. Consequently if the given curve is a parabola, the parameterization (3.6) is polynomial only when (3.7) is satisfied by the point (X_4, Y_4, Z_4) on the parabola.

4. Two-dimensional nonlinear transformations

We want to construct a quadratic rational transformation that will map the standard triangle (Fig. 2(a)) onto the element with two straight sides and one curved side with parameterization

(3.6) (Fig. 2(c)). For the transformation to be conforming and to match adjacent elements without any discontinuity of interpolant, we require the axes to remain pointwise invariant under the transformation, that is,

$$\begin{aligned} \text{image}((0, q)) &= (0, qZ, Z) \text{ for all } q, \\ \text{image}((p, 0)) &= (pZ, 0, Z) \text{ for all } p. \end{aligned} \quad (4.1)$$

The transformation therefore has the general form

$$\left. \begin{aligned} X &= p(a_1 + a_2 p + a_3 q), \\ Y &= q(a_1 + b_2 p + b_3 q), \\ Z &= a_1 + a_2 p + b_3 q + c_4 pq. \end{aligned} \right\} \quad (4.2)$$

Since we furthermore require (3.6) to be the image of (2.3) under the transformation (4.2), we replace $p = t$, $q = 1 - t$, in (4.2) and compare the result with (3.6), to obtain

$$\left. \begin{aligned} a_1 &= Z_t - f, & a_2 &= f - \beta, & a_3 &= 2X_t - Z_t + f, \\ b_2 &= 2Y_t - Z_t + f, & b_3 &= f - \alpha, & c_4 &= \alpha + \beta, \end{aligned} \right\} \quad (4.3)$$

where α and β are defined by

$$c\alpha = c(2Y_t + Z_t) - 2kY_4, \quad c\beta = c(2X_t + Z_t) - 2kX_4, \quad (4.4)$$

f is arbitrary and $\alpha = \beta = 0$ for (3.7) to be satisfied, that is, for parabolae to have polynomial parameterizations. Since we should like (4.2) to revert to the quadratic isoparametric transformations for given parabolic curved edges, we may require f to be a combination of α and β , namely

$$f = f_1\alpha + f_2\beta, \quad (4.5)$$

where f_1 and f_2 are arbitrary constants.

The freedom present in the construction of the transformation, which is represented by the constants f_1 and f_2 , occurs as a result of the transition from polynomials in one variable to those in two variables. The amount of freedom will depend on the number of curves to be matched, and could conceivably be used to ensure bijectivity. This aspect, which still has to be investigated, will involve establishing conditions under which the Jacobian of the transformation does not vanish on the element.

5. Basis construction

As was the case for the construction of a suitable transformation described in the previous section, as well as in the curve parameterization, freedom of choice also occurs when we simulate isoparametric transformations (1.3) by introducing basis functions in (4.2). These basis functions should in general be rational and at least of degree two, but many different sets can yield the same transformation.

We require the basis, as in the case of Lagrange interpolation, to be biorthonormal with respect to evaluation at selected nodes on the standard triangle, to be linearly independent and therefore to yield unique interpolants, and to satisfy (1.2) and (1.3). Note that in Cartesian coordinates the transformation (4.2) has the form

$$x = p\theta(p, q)/\chi(p, q), \quad y = q\phi(p, q)/\chi(p, q), \quad (5.1)$$

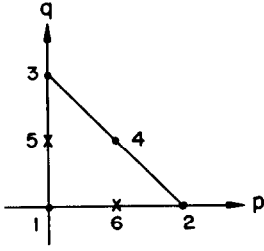


Fig. 3. Node configurations for basis construction.

where $\theta(p, q)$ and $\phi(p, q)$ are linear and $\chi(p, q)$ is quadratic. Since the curved edge is second order, we need at least three nodes on the hypotenuse for unique interpolation. We illustrate the construction of a basis with two examples.

Example 1. Linear interpolants on the axes. The simplest possible case requires nodes 1 to 4 (Fig. 3), where for simplicity we assume node 4 to be at $(\frac{1}{2}, \frac{1}{2})$. For (1.3) to be satisfied we need

$$x = W_2 + (X_4/Z_4)W_4, \quad y = W_3 + (Y_4/Z_4)W_4. \quad (5.2)$$

Since each basis function is unique and is zero at all the nodes not associated with it, it must be identically zero on edges not containing the associated node, that is,

$$W_2(0, q) = 0, \quad W_3(p, 0) = 0, \quad W_1(p, 1-p) = 0, \quad W_4(p, 0) = W_4(0, q) = 0.$$

If we choose the lowest possible degrees for the basis functions, we must have

$$W_4 = 4\chi(\tfrac{1}{2}, \tfrac{1}{2}) \frac{pq}{\chi(p, q)} = 2\frac{k}{c} Z_4 \frac{pq}{\chi(p, q)},$$

so that, from (5.1) and (5.2),

$$W_2 = \frac{p(\theta(p, q) - 2(k/c)X_4q)}{\chi(p, q)}, \quad W_3 = \frac{q(\phi(p, q) - 2(k/c)Y_4p)}{\chi(p, q)},$$

and, finally, from (1.2),

$$W_1 = 1 - W_2 - W_3 - W_4.$$

That $W_1 = 0$ on $1 - p - q = 0$ follows from the fact that its numerator is quadratic and is zero at three points on a line—it must therefore vanish identically on the line, from Bezout's theorem [3]. Linear independence follows similarly from the fact that the interpolant of zero nodal values must be identically zero on the element. The linearity of the interpolants on the axes follows from (5.2), since W_4 vanishes and x and y are linear, so that all the basis functions are linear on the axes. For parabolic images of the hypotenuse and polynomial transformations ($\alpha = \beta = 0$) the basis is bilinear.

Example 2. Quadratic interpolants on the axes. We introduce two midside nodes 5 and 6 on the axes (Fig. 3). For (1.3) to be satisfied we need

$$x = \tfrac{1}{2}W_6 + W_2 + (X_4/Z_4)W_4, \quad y = \tfrac{1}{2}W_5 + W_3 + (Y_4/Z_4)W_4. \quad (5.3)$$

Once again choosing the simplest forms for the basis functions and taking into account that each must vanish identically on edges not containing its associated node, we obtain the basis functions

$$W_4 = 4\chi\left(\frac{1}{2}, \frac{1}{2}\right) \frac{pq}{\chi(p, q)} = 2(k/c)Z_4 \frac{pq}{\chi(p, q)},$$

$$W_5 = 4\chi\left(0, \frac{1}{2}\right) \frac{q(1-p-q)}{\chi(p, q)}, \quad W_6 = 4\chi\left(\frac{1}{2}, 0\right) \frac{p(1-p-q)}{\chi(p, q)}.$$

The basis functions W_2 and W_3 can now be found from (5.1) and (5.3), while W_1 once again follows from (1.2). Biorthonormality and linear independence follow in the same way as for the previous example. For $\alpha = \beta = 0$ the basis reverts to the usual second order basis for the standard triangle [8].

Note that care should be taken that the hyperbola $\chi(p, q) = 0$ does not intersect or touch the edges of the triangle. In a more sophisticated approach this would correspond to imposing constraints on the free parameters.

6. Conclusion

We have shown that it is possible to construct transformations similar to the isoparametric transformations for triangular elements with one conic edge. These transformations are in general rational and quadratic. The procedure allows us to perform required calculations with respect to the geometrically simpler master-elements. In the particular case where the given conic is a parabola, the transformations revert to the quadratic isoparametric transformation. Extensions of the method, both to systematic parameterization of higher order curves and to parametric transformations for elements with different shapes, will be considered in the future.

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